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Fractals and Chaos in Cancer Models

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We argue that tumor growth, considered as a dynamical system, is chaotic. Chaotic models are proposed which fit the observations well. Some of these models treat the tumor as a fractal.

One of the established facts in the study of malignant tumors is that they differ markedly from patient to patient even if they are of the same type (Wheldon, 1988). Furthermore tumor growth is not identical even in the same patient. If we consider tumor growth as a dynamical system, this property is known as sensitivity to initial conditions (Devany, 1989). This is synonymous to chaos (Devany, 1989). Therefore it is suitable to use chaotic models to describe the growth of malignant tumors.

The standard model used is the Gompertz equation (Swan, 1990)

$$1/N\ dN/dt = a\ \ln(N_0/N) \tag{1}$$

where $N(t)$ is the number of cells of the tumor. The Gompertz model fits well many human tumors in the observed region $N(t) \geq 10^9$ cells. The corresponding finite-difference equation is

$$N_{t+1} = N_t + aN_t[\ln(N_*/N_t)] \tag{2}$$

It is straightforward to show that the nonzero steady-state solution of (2) is stable if $0 < a < 2$.

The disadvantage of the Gompertz law is that it is not biologically motivated. Recently a biologically motivated model has been proposed (Ahmed, 1990). The tumor is assumed to consist of a core and a surface. The cell growth is proportional to the surface and the cell loss to the volume.

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The dynamical equation is therefore

$$dN/dt = -aN(t) + b[N(t)]^{2/3} \tag{3}$$

The corresponding finite-difference equation is

$$N_{t+1} = (1-a)N_t + b[N_t]^{2/3} \tag{4}$$

It has been shown that for a suitable choice for a and b the growth described by (3) is very close to that of the Gompertz equation in the observable region.

We show that the system (4) is chaotic. Rescaling (4), one gets the recursion

$$N_{t+1} = m(N_t^{2/3} - N_t) \tag{5}$$

The nonzero steady-state solution of (5) is

$$N = [m/(m+1)]^3 \tag{6}$$

and numerical calculations have shown that this solution is stable if $5 > m > 0$. For $6.1 > m > 5$, the system (5) has a 2-cycle ($N_{t+2} = N_t$). For $6.3 > m > 6.1$, it has a 4-cycle. For $6.4 > m > 6.3$, it has a 16-cycle. For $m > 6.4$, it is chaotic.

A point that deserves further discussion is that in deriving (3) and (4) it has been assumed that the surface area A is proportional to its volume to the power 2/3,

$$A \propto V^{2/3} \tag{7}$$

However, the tumor may be approximated better as a fractal, for which

$$A \propto V^{d_f}, \qquad 0 < d_f < 1 \tag{8}$$

This will modify the models (3) and (4) to the form

$$dN(t)/dt = -aN(t) + b[N(t)]^{d_f} \tag{9}$$

$$N_{t+1} = (1-a)N_t + b[N_t]^{d_f} \tag{10}$$

It is expected that one can choose a, b, d_f to fit the observations (as has been done in the case $d_f = 2/3$). Thus it remains to show that the system (10) is chaotic. Rescaling, one gets

$$N_{t+1} = m[(N_t)^{d_f} - N_t] \tag{11}$$

Define the function $gm(x)$ by

$$gm(x) = m[(x)^{d_f} - x], \qquad 0 < d_f < 1 \tag{12}$$

The nonzero steady-state solution for $gm(x)$ is

$$p=[m/(m+1)]^{1/(1-d_f)} \tag{13}$$

Thus $0<p<1$ for all $m<0$. Numerical calculations show that the function $gm(x)$ is chaotic for the value

$$m>1/[(d_f)^{d_f/(1-d_f)}-(d_f)^{1/(1-d_f)}] \tag{14}$$

except for $x=0$ and $x=1$.

Therefore from both geometrical and dynamical points of view the models (9) and (10) are more suitable to describe tumor growth. It is interesting that deep dynamical and geometrical concepts like chaos and fractals plays a role in medical problems (West, 1990).

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